Computer Science Department

TECHNICAL REPORT

"Decision Procedures for Elementary Sublanguages of Set Theory.
VII. Validity in Set Theory When a Choice Operator is Present"

by

A. Ferro

and

E. G. Omodeo

Technical Report #226
June, 1986

NEW YORK UNIVERSITY

Department of Computer Science
Courant Institute of Mathematical Sciences
251 MERCER STREET, NEW YORK, N.Y. 10012

"Decision Procedures for Elementary Sublanguages
of Set Theory.
VII. Validity in Set Theory When a Choice
Operator is Present"

by

A. Ferro

and

E. G. Omodeo

Technical Report #226

June, 1986

Decision Procedures for Elementary Sublanguages of Set Theory.
VII. Valldity in Set Theory When a Choice Operator is Present

by

A. Ferro
University of Catania and Courant Institute
and
E.G. Omodeo
Enidata - Bologna

1. Introduction

In a language designed to formalize set theory, it is useful to incorporate a primitive unary operator η about which one postulates that $\eta s \in s$ and $\eta s \cap s = \phi$ for every nonempty set s. The existence of such an η implies the validity of both the foundation (or regularity) axiom and the axiom of choice (see [8], [9], [10]). Moreover, in automatic theorem proving the introduction of such an operator seems to be the most natural way of formalizing induction proofs in mathematics (cf. [11]).

In this paper we consider an unquantified theory whose only relators and operators are $=$ (equality), $\in$ (membership), $\bigcup$ (binary union), $\setminus$ (set difference), η ; in addition to these symbols, our language involves the usual propositional connectives and denumerably many variables, which are supposed to range over *finite* sets. (Apart from the restriction of considering only finite sets this is an extension of the theory MLS considered in [1]). We show that if we impose some further semantical constraints on η, then for every formula p of our language either p is valid, i.e. true under all possible assignments of finite sets to the terms, or some of these assignments makes p false, independently of the particular choice of η. We achieve this by giving an algorithm which either detects the validity of p or builds a counterexample of p (independent of p) when p is invalid.

We consider also the language in which the symbols $\bigcup$, $\setminus$ are not allowed (cf. [2]) but in which variables are supposed to range over *arbitrary* sets. A similar completeness theorem

is proven under weaker assumptions on η by giving alternative decision method applicable only to formulae of this sublanguage.

2. Semantics of the Choice Operator η

If we simply required that $\eta s \in s$ and $\eta s \bigcap s = \phi$ for every nonempty set s, and assigned some default value to $\eta\phi$, then the usual relationship between validity and satisfiability can not be made independent of the particular choice of the function η. For example the formula

$$\eta x \in y \& \eta y \in x \rightarrow \eta x = \eta y \tag{2.1}$$

neither is valid, i.e. true under all possible assignments of (finite) sets to variables, nor its negation has a model independent of η. The same holds for the formulas

$$v \in w \& w \in \eta x \rightarrow v \notin x \tag{2.2}$$

$$v \subseteq \eta x \& v \neq \eta x \rightarrow v \notin x \tag{2.3}$$

To avoid this kind of problem we will put more semantical constraints on η. More precisely we assume that for some well ordering $<$ of all sets the following restrictions are satisfied:

R_0 $\quad \eta\phi = \phi$ (*empty restriction*)

R_1 $\quad x \neq \phi \rightarrow \eta x \in x$ (*choice restriction*)

R_2 $\quad y \in x \rightarrow \eta x \leq y$ (*minimality restriction*)

R_3 $\quad y \in x \rightarrow y < x$ (*regularity restriction*)

R_4 $\quad \{x_1, x_2, \ldots, x_n\} \subsetneq x \rightarrow \{x_1, x_2, \cdots x_n\} < x$ (*finite monotonicity restriction*)

R_5 $x_1, \ldots, x_n < y_0 < y_1, \ldots, y_m \rightarrow \{x_1, \cdots, x_n, y_1, \ldots, y_m\} < \{y_0, y_1, \ldots, y_m\}$ (*antilexicographic restriction)*

From R_5 it follows immediately the following

LEMMA 2.1. If A,B are *finite* sets then either

(i) $A \subseteq B$ and $A \leq B$, or

(ii) $B \subseteq A$ and $B \leq A$, or

(iii) $A<B$ if and only if $\max(A \backslash B) < \max(B \backslash A)$.

An immediate consequence is the following

R_6. If x,y,z are *finite* sets then

$$x<y \,\&\, (x \bigcup y) \bigcap z = \phi \rightarrow (x \bigcup z) < (y \bigcup z)$$

Moreover the following is also true.

R_7 Let $A_1<A_2< \cdots <A_n$ be nonempty *finite* sets which are pairwise disjoint then

$$A_{i_1} \bigcup A_{i_2} \bigcup \cdots \bigcup A_{i_k} < A_{j_1} \bigcup A_{j_2} \bigcup \cdots \bigcup A_{j_m}$$

if and only if

$$\{i_1,i_2,\ldots,i_k\}<\{j_1,j_2,\ldots,j_m\}$$

in the antilexicographic ordering.

Indeed assume that $i_1<i_2<\ldots<i_k$ and $j_1<j_2< \cdots <j_m$. Clearly

$$\{i_1,i_2,\ldots,i_k\} \subseteq \{j_1,j_2,\ldots,j_m\} \leftrightarrow A_{i_1} \bigcup A_{i_2} \bigcup \cdots \bigcup A_{i_k} \subseteq A_{j_1} \bigcup A_{j_2} \bigcup \cdots \bigcup A_{j_m} \tag{2.4}$$

and

$$\{j_1,j_2,\ldots,j_m\} \subseteq \{i_1,i_2,\ldots,i_k\} \leftrightarrow A_{j_1} \bigcup A_{j_2} \bigcup \cdots \bigcup A_{j_m} \subseteq A_{i_1} \bigcup A_{i_2} \bigcup \cdots \bigcup A_{i_k}. \tag{2.5}$$

Therefore if one of (2.4) and (2.5) holds then R_7 is plain. Otherwise put $I=\{i_1,i_2,\ldots,i_k\}$, $J=\{j_1,j_2,\ldots,j_m\}$. By R_6 and by Lemma 2.1 we get

$$\bigcup_{i \in I} A_i < \bigcup_{j \in J} A_j \leftrightarrow \bigcup_{i \in I \backslash J} A_i < \bigcup_{j \in J \backslash I} A_j$$

$$\leftrightarrow \max\Big(\bigcup_{i \in I \backslash J} A_i\Big) < \max\Big(\bigcup_{j \in J \backslash I} A_j\Big)$$

$$\leftrightarrow \max A_{\max(I\setminus J)} < \max A_{\max(J\setminus I)}$$

$$\leftrightarrow A_{\max(I\setminus J)} < A_{\max(J\setminus I)}$$

$$\leftrightarrow \max(I\setminus J) < \max(J\setminus I)$$

$$\leftrightarrow I < J$$

which completes the proof of R_7.

3. Consistency

In this section we show the existence of a function η satisfying restrictions $R_0 - R_5$ (and hence also R_6 and R_7). To this end we consider the Von Neumann hierarchy of all sets.

$$V_0 = \phi$$

$$V_{\alpha+1} = \{s \mid s \subseteq V_\alpha\} \text{ for } \textit{every ordinal } \alpha$$

$$V_\beta = \bigcup_{\gamma \in \beta} V_\gamma \text{ for } \textit{every limit ordinal } \beta .$$

It is well known that we can consistently assume that for every set s there is an ordinal α such that $s \subseteq V_\alpha$; the minimum such ordinal is called the *rank* of s and is written *rank s*. We define a well ordering of all sets in the following way:

We first put $s<t$ whenever rank $s<$ rank t. To order sets having the same rank α we proceed by induction on α. Indeed there is only one set of rank zero, namely the empty set. Next assume we have ordered all sets of rank less than α and let s and t be two sets of rank α. If s and t are both infinite we put $s<t$, where $<$ is any well ordering of all infinite sets of rank α. If one is finite and the other is infinite then we make the finite set preceding the infinite set. Finally if they are both finite we order them antilexicographically (this makes sense, because by the induction hypothesis the elements of s and t have been already ordered). This completes the definition of a well ordering "$<$" of all sets. Now put

$$\eta\phi = \phi \text{ and } \eta s = \textit{the least element of } s \textit{ with respect } <$$

It is immediate to verify that restrictions R_0,R_1 and R_2 are satisfied. Moreover if $y \in x$ then rank $(y) <$ rank (x). This shows that R_3 is also satisfied. Furthermore if $\{x_1, x_2, \ldots, x_n\} \subsetneq x$,

$rank\ \{x_1,x_2,\ldots,x_m\} = rank\ x$ and x is finite then by the antilexicographic property $\{x_1,x_2,\ldots,x_n\} < x$. This yields R_4. Finally if $rank\ (x_i) \leq rank\ (y_0), i=1,\ldots,n$, then $rank\ \{x_0,x_1,\ldots,x_n,y_1,\ldots,y_m\} \leq rank\ \{y_0,y_1,\ldots,y_m\}$. This shows that R_5 holds completing the proof of the existence of a function η satisfying all the conditions $R_0 - R_5$.

4. Preliminaries on Multilevel Syllogistic

Multilevel Syllogistic (abbreviate MLS) is the unquantified theory whose language consists of

variables $x,y,z,\ldots$,

the operators $\bigcup, \bigcap, \setminus$,

the relators $\in, =$.

In addition to these symbols we can use the usual propositional connectives $\neg, \&, V, \rightarrow, \leftrightarrow$. Variables are supposed to range over arbitrary sets whereas the operators and relators are interpreted in the usual set-theoretical sense. An example of a formula in MLS is the following:

$$x \in (y \bigcup z) \& x \notin y \rightarrow x \in z.$$

This theory was shown to be decidable in [1]. The method described in [1] can be rephrased as follows. First we can limit ourselves, without loss of generality, to show how to test satisfiability of any finite conjunction Q of literals of the following type:

$$(=) x = y \bigcup z, x = y \setminus z$$

$$(\in, \notin) x \in y, x \notin y$$

To describe how to accomplish this we need some definitions.

DEFINITION 4.1. A place α of Q is a $\phi/\{\phi\}$ -valued function defined on the variables appearing in Q and which satisfies all literals of type $(=)$ in Q.

Notice that there are only finitely many places of Q.

DEFINITION 4.2. A place α at x of Q is a place of Q such that $\alpha(y)=\{\phi\}$ (resp. ϕ) whenever $x \in y$ (resp. $x \notin y$) appears in Q.

Let $\sim$ be the equivalence relation defined by $x \sim y \leftrightarrow \alpha(x)=\alpha(y)$ for every place α of Q. Partition the variables of Q into equivalence classes, pick a representative $\bar{x}$ in each class. $\{y : y \sim x\}$ and replace each variable x in Q by its representative $\bar{x}$. Let $\bar{Q}$ be the resulting conjunction and let $Y=\{y_1,y_2,\ldots,y_m\}$ be the set of all variables of $\bar{Q}$.

DEFINITION 4.3. Let Γ be a set of places of $\bar{Q}$. Then $x \sim_{\Gamma} y$ will be an abbreviation for $(\forall \alpha \in \Gamma)(\alpha(x)=\alpha(y))$. The following states the decidability of MLS [1].

THEOREM 4.1 *Q has a model if and only if there is a set $\Gamma=\{\alpha_1,\alpha_2, \cdots ,\alpha_n\}$ of pairwise distinct places of $\bar{Q}$, an ordering $<$ of $Y/\sim_{\Gamma}$, and a function $F:\{1, \cdots ,m\} \to \{1, \cdots ,n\}$ such that:*

$$\alpha_{F(i)} \text{ is a place at } y_i \text{ of } \bar{Q} \text{ for every } i=1,2, \cdots ,m \tag{4.1}$$

$$y_i \sim_{\Gamma} y_j \to F(i)=F(j) \tag{4.2}$$

$$\alpha_{F(j)}(y_i)=\{\phi\} \to \bar{y}_i > \bar{y}_j \tag{4.3}$$

(*where $\bar{y}$ denotes the element of $Y/\sim_{\Gamma}$ containing y*).

If $\Gamma,<,F$ exist in such a way as to satisfy conditions (4.1) - (4.3) then models of Q can be built as follows. Choose sets $\sigma_j, j=1,\ldots,n$. Defining My_h before My_k whenever $\bar{y}_h<\bar{y}_k$ put:

$$My_i = \bigcup_{\alpha_j(y_i)=\{\phi\}} \sigma_j \bigcup \left\{ My_k : \alpha_{F(k)}(y_i)=\{\phi\} \,\&\, 1 \le k \le m \right\}. \tag{4.4}$$

Complete the definition of M by putting $Mx=M\bar{x}$ for any other variable of Q. Then we have the following basic fact [1]:

THEOREM 4.2. *Formula* (4.4) *defines a model of Q whenever the following conditions are satisfied*

$$\sigma_i \bigcap \sigma_j = \phi \textit{ whenever } i \neq j \tag{4.5}$$

$$My_i \notin \sigma_j \text{ for } \textit{every } i = 1,\ldots,m \text{ and } j = 1,\ldots,n \tag{4.6}$$

$$\sigma_j \neq \phi \textit{ unless } j = F(k) \text{ for } \textit{some } k. \tag{4.7}$$

5. Decidability of Finite Satisfiability for MLS Extended with a Choice Operator

We extend the language of MLS by adding a new unary operator η and show that the following completeness result holds for this extended theory which we call MLSη.

THEOREM 5.1: *For every formula* ψ *of* MLSη *either* ψ *is true under all finite interpretations (i.e. interpretations in which the value of each term in* ψ *is a finite set) or its negation is satisfied by some finite interpretation independently of the particular choice of* η *satisfying restrictions* $R_0 - R_5$.

We prove this theorem by giving an algorithm which decides if $\neg\psi$ has or not finite models and in the positive case is able to construct a finite model of $\neg\psi$ which is independent of the particular interpretation of η. By the very same argument used in MLS we can restrict ourselves to consider a finite conjunction Q_η of literals of the following types:

$$(=) \quad x = y \bigcup z, \quad x = y \backslash z$$

$$(\in, \notin) \quad x \notin y, \quad x \notin y$$

$$(\eta) x \ = \ \eta y$$

Let Q be the set of statements of type $(=),(\in,\notin)$ *in* Q_η and let $\bar{Q}_\eta$ be the result of replacing in Q_η each variable by its representative in the equivalence relation determined by all places of Q. Let $Y = \{y_1, y_2, \ldots, y_m\}$ be the set of all the variables appearing in Q_η. Our main result is a consequence of the following decidability theorem.

THEOREM 5.2 Q_η *has a finite model if and only if there exist a set of pairwise distinct places* $\Gamma = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of $\bar{Q}$ *and a function* $F:\{1, \ldots, m\} \to \{1, \ldots, n\}$ *such that*

$$\alpha_{F(i)} \textit{ is a place at } y_i \textit{ of } \bar{Q} \text{ for } \textit{every } i = 1,2,\ldots,m. \tag{5.1}$$

$$\alpha_{F(i)}(y_k) \neq \alpha_{F(j)}(y_k) \to \exists \alpha_l \in \Gamma \textit{ such that } \alpha_l(y_i) \neq \alpha_l(y_j) \tag{5.2}$$

Moreover, let $<$ *be the ordering on* $Y/\sim_\Gamma$ *defined as follows. Put*

$$\Delta_i = \{j : \alpha_j(y_i) = 1\}$$

and let $\bar{y}_i < \bar{y}_j$ *whenever* Δ_i *precedes* Δ_j *in the antilexicographic ordering of finite sets of integers. Then the following properties must also hold*

$$\alpha_{F(j)}(y_i) = \{\phi\} \to \bar{y}_j < \bar{y}_i \tag{5.3}$$

$$\alpha_j(y_i) = \{\phi\} \to j < F(i) \tag{5.4}$$

If $y_{j^*} = \eta y_j$ *appears in* $\bar{Q}_\eta$ *then either* $\alpha_k(y_j) = \phi$ for *all* $k = 1,2,..,n$ and $y_j \sim_\Gamma y_{j^*}$ *or* (5.5)

$$\alpha_{F(j^*)}(y_j) = \{\phi\} \text{ and}$$

$$\alpha_{F(k)}(y_j) = \{\phi\} \to (y_{j^*} \sim_\Gamma y_k V \bar{y}_{j^*} < \bar{y}_k) \tag{5.5.a}$$

$$\alpha_t(y_j) = \{\phi\} \to \forall\, \alpha_\ell \in \Gamma(\alpha_\ell(y_{j^*}) = \{\phi\} \to l < t) \tag{5.5.b}$$

Furthermore if conditions (5.1) - (5.5) are all satisfied then a finite model of Q_η *, independent of the particular choice of* η *, can be effectively constructed.*

Proof. Assume that Q_η has a finite model M and let $My_{i_1}, \ldots, My_{i_k}$ with $1 \le i_j \le m$ be pairwise distinct sets such that $\{My_{i_1}, \ldots, My_{i_k}\} = \{My_1, \ldots, My_m\}$. Let $A_1, A_2, \ldots, A_n$ be the nonempty disjoint parts of the Venn diagram determined by $My_{i_1}, \ldots, My_{i_k}$. Assume $A_1 < A_2 < \cdots < A_n$ in the well ordering of sets associated with η. Let $\alpha_1, \ldots, \alpha_n$ be the places of $\bar{Q}$ defined by

$$\alpha_i(y_j) = \{\phi\} \text{ if and } only \text{ if } A_i \subseteq My_j.$$

Put $F(i) = k$ if and only if $My_i \in A_k$. We claim that $\Gamma = \{\alpha_1, \ldots, \alpha_n\}$ and F satisfy conditions (5.1) - (5.5). Indeed conditions (5.1) and (5.2) are immediate. To verify the remaining conditions we will make use of the following

LEMMA 5.3.

$$y_i \sim_\Gamma y_j \text{ if and } only \text{ if } My_i = My_j \tag{5.6}$$

$\bar{y}_i < \bar{y}_j$ if and *only* if $My_i < My_j$ *in the well ordering of all sets associated with* η. (5.7)

Proof. (5.6) is plain by the definition of the places α_i. Moreover condition (5.7) is an immediate consequence of R_7 proven in Section 2.

To verify condition (5.3), assume that $\alpha_{F(j)}(y_i)=\{\phi\}$ then $My_j \in My_i$ and by R_3 it follows $My_j < My_i$. Applying Lemma 5.3 we get $\bar{y}_j < \bar{y}_i$ which proves (5.3). Next we verify (5.4). Let $\alpha_j(y_i)=\{\phi\}$, then $A_j \subseteq My_i \in A_{F(i)}$. By R_4 and R_3 we have $A_j < A_{F(i)}$ and hence $j < F(i)$ which completes the verification of (5.4). Finally to show that (5.5) also holds assume that $y_{j^*} = \eta y_j$ appears in $\bar{Q}_\eta$. Since M is a model of Q_η then $My_{j^*} = \eta My_j$. Thus if $My_j = \phi$ then $My_{j^*} = \phi$ and by Lemma 5.3 $y_{j^*} \sim_\Gamma y_j$. Otherwise if $My_j \neq \phi$ then by R_1 $My_{j^*} \in My_j$ and $\alpha_{F(j^*)}(y_j) = \{\phi\}$. Therefore if $\alpha_{F(k)}(y_j) = \{\phi\}$ for some k then $My_k \in My_j$. By R_2 it follows that $My_{j^*} \leq My_k$ and hence by Lemma 5.3 either $y_{j^*} \sim_\Gamma y_k$ or $\bar{y}_{j^*} < \bar{y}_k$ showing (5.5.a). Moreover if $\alpha_t(y_j) = \{\phi\}$ then $\phi \neq A_t \subseteq My_j$. If follows by R_2 that My_{j^*} is less than or equal to each element of A_t. By applying R_3 we have $My_{j^*} < A_t$. Therefore if $\alpha_\ell(y_{j^*}) = \{\phi\}$ then $A_\ell \subseteq My_{j^*} < A_t$ which by R_4 gives $A_\ell < A_t$. This yields $l < t$ completing the proof of (5.5 b) and showing that Theorem 5.2 holds in one direction.

Conversely, assume that $\Gamma = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and F can be found in such a way as to satisfy all the conditions (5.1) - (5.5). We will show how to build a finite model of Q_η which is independent of the particular choice of η (subject only to the restrictions $R_0 - R_5$).

Let I be a finite set of *odd* rank and put

$$I_1 = I$$

$$I_m = \{\{I_{m-1}\}\} \text{ for } m > 1$$

Notice that all these sets I_m have odd ranks r_m and $r_1 < r_2 < \cdots$ Moreover $I_1 < I_2 < \cdots$ in any well ordering of all sets satisfying restriction R_3.

Next put

$$\sigma_j = \{I_j\} \text{ for } every\ j = 1,2 \ldots, n \tag{5.8}$$

Defining My_i before My_j whenever $\bar{y}_i < \bar{y}_j$ put:

$$My_i = \bigcup_{\alpha_j(y_i)=\{\phi\}} \sigma_j \bigcup \{My_k : \alpha_{F(k)}(y_i) = \{\phi\}\} \tag{5.9}$$

Extend the definition of M to every variable y of Q_η by putting $My = M\bar{y}$. Since by hypothesis conditions (4.1), (4.2), and (4.3) of Theorem 4.1 are satisfied then we can try to apply Theorem 4.2. Since conditions (4.5) and (4.7) are trivially satisfied by (5.8) then it remains only to verify that (4.6) also holds. To this end we make use of the following

LEMMA 5.4

$$rank\ (My_i) = \max\ \{rank(\sigma_j) : \alpha_j(y_i) = \{\phi\}\}$$

Proof. We proceed by induction. If $\bar{y}_i$ is minimum in $Y/\sim_\Gamma$ with respect to the ordering defined in the statement of Theorem 5.2, then by (5.3) and (5.9)

$$My_i = \bigcup_{\alpha_j(y_i)=\{\phi\}} \sigma_j$$

and the lemma is trivial.

Next assume that the lemma holds for every y_j less than y_i. Let

$$p = rank\ (\bigcup_{\alpha_j(y_i)=\{\phi\}} \sigma_j) = \max\ \{rank\ \sigma_j : \alpha_j(y_i) = \{\phi\}\ \}$$

and let

$$q = rank\{My_k : \alpha_{F(k)}(y_i) = \{\phi\}\} = \max\ \{\ rank\ (My_k) : \alpha_{F(k)}(y_i) = \{\phi\}\} + 1$$

We want to show that $p \geq q$. Indeed by the induction hypothesis and by (5.3) if $\alpha_{F(k)}(y_i) = \{\phi\}$, then rank $(My_k) = \max\ \{\ rank\ (\sigma_j) : \alpha_j(y_k) = \{\phi\}\}$. But by (5.4) if $\alpha_j(y_k) = \{\phi\}$ then $j < F(k)$ and hence rank $(\sigma_j) <$ *rank* $(\sigma_{F(k)})$. It follows that rank $(My_k) <$ *rank* $(\sigma_{F(k)})$ for every k such that $\alpha_{F(k)}(y_i) = \{\phi\}$. Therefore

$$q - 1 < \max\{\ rank\ (\sigma_{F(k)}) : \alpha_{F(k)}(y_i) = \{\phi\}\} \leq p.$$

This yields $q \leq p$ completing the proof of the lemma.

By this lemma it follows that each My_i has an *even* rank whereas each I_j has an odd rank. This implies $My_i \notin \sigma_j$, proving (4.6). Then we can apply Theorem 4.2 having that (5.9) defines a model M of all MLS statements in Q_η. In order to verify that M is indeed a model of Q_η no matter how η is chosen we proceed as follows. Assume that $y = \eta x$ appears

in Q_η and let $y_{j*}=\bar{y}$ and $y_j=\bar{x}$. Thus $y_{j*}=\eta y_j$ appears in $\bar{Q}_\eta$. By (5.5) if $\alpha_k(y_j)=\phi$ for all k then $y_{j*}\sim_\Gamma y_j$, $My_{j*}=My_j=\phi=Mx=My$ and $My=\eta Mx$. Otherwise $\alpha_{F(j*)}(y_j)=\{\phi\}$ which yields $My_{j*}\in My_j$. We will show that My_{j*} is indeed the least element of My_j in any well ordering of all sets satisfying R_3, R_4, and R_5.

To this end we show the following lemma in which $<$ is the ordering of $Y/\sim_\Gamma$ mentioned in Theorem 5.2.

LEMMA 5.5. If $\bar{y}_h<\bar{y}_k$ *then* $My_h<^*My_k$ in any well ordering $<^*$ of all sets satisfying R_3,R_4 and R_5.

To prove this lemma we need to show that the following is also true.

LEMMA 5.6.

$$My_h<^*I_{F(h)}$$

in any well ordering $<^*$ *of all sets satisfying* R_3,R_4, and R_5

Proof: Again we proceed by induction. If $\bar{y}_h$ is the least element of $Y/\sim_\Gamma$ with respect to the ordering $<$ defined in the statement of Theorem 5.2 then $My_h=\bigcup_{\alpha_t(y_h)=\{\phi\}}\sigma_t$ by (5.3) and (5.9). But by (5.4) if $\alpha_t(y_h)=\{\phi\}$ then $t<F(h)$ so that $I_t\leq^*I_{F(h)-1}<^*\{I_{F(h)-1}\}$ in any well ordering $<^*$ of sets satisfying R_3,R_4, and R_5. By R_5 we get

$$My_h=\{I_t:\alpha_t(y_h)=\{\phi\}\}<^*\{\{I_{F(h)-1}\}\}=I_{F(h)}$$

proving our lemma when $\bar{y}_h$ is the least element of $Y/\sim_\Gamma$.

Next assume that the lemma holds for every y_k with $\bar{y}_k<\bar{y}_h$. By (5.8) and (5.9) we have that $My_h=\{I_t:\alpha_t(y_h)=\{\phi\}\}\bigcup\{My_k:\alpha_{F(k)}(y_h)=\{\phi\}\}$. Now if $\alpha_t(y_h)=\{\phi\}$ then by (5.4) $t<F(h)$ and $I_t\leq^*I_{F(h)-1}<^*\{I_{F(h)-1}\}$. Moreover if $\alpha_{F(k)}(y_h)=\{\phi\}$ then by the induction hypothesis $My_k<^*I_{F(k)}$ and by (5.4) $F(k)<F(h)$. Thus $My_k<^*I_{F(k)}\leq^*I_{F(h)-1}<^*\{I_{F(h)-1}\}$. We can then conclude that every element of My_h is less than $\{I_{F(h)-1}\}$ in any well ordering $<^*$ of all sets

satisfying R_3-R_5. This implies that $My_h <^* \{\{I_{F(h)-1}\}\} = I_{F(h)}$ completing the proof of Lemma 5.6.

Now we are ready to prove Lemma 5.5. Indeed if $\bar{y}_h < \bar{y}_k$ then $\Delta_h < \Delta_k$ in the anti-lexicographic ordering of all sets of integers. Let $\Delta_h = \{i_1, i_2, \ldots, i_t\}$ and $\Delta_k = \{j_1, j_2, \ldots, j_m\}$ with $i_1 < i_2 < \cdots < i_t$ and $j_1 < j_2 < \cdots < j_m$. Since $\Delta_h < \Delta_k$ then there exists $j_l \in \{j_1, \ldots, j_m\}$ such that $\{j_{\ell+1}, \ldots, j_m\} \subseteq \{i_1, \ldots, i_t\}$ and $\{i_1, \ldots, i_t\} \setminus \{j_1, \cdots, j_m\} < \{j_\ell\}$. Thus

$$\bigcup_{\substack{\alpha_r(y_h)=\{\phi\}\& \\ \alpha_r(y_k)=\phi}} \sigma_r <^* \sigma_{j_\ell}$$

On the other hand if $\alpha_{F(q)}(y_h) = \{\phi\} \& \ \alpha_{F(q)}(y_k) = \phi$ then $F(q) < j_\ell$ and by Lemma 5.6 $My_q <^* I_{F(q)} <^* I_{j_\ell}$. Therefore we have that

$$\bigcup_{\substack{\alpha_r(y_h)=\{\phi\}\& \\ \alpha_r(y_k)=\phi}} \sigma_r \bigcup \{My_q : \alpha_{F(q)}(y_h) = \{\phi\} \& \alpha_{F(q)}(y_k) = \phi\} <^* \sigma_{j_\ell}$$

Using R_6 and R_4 we get

$$My_h = \bigcup_{\alpha_r(y_h)=\{\phi\}} \sigma_r \bigcup \{My_q : \alpha_{F(q)}(y_h) = \{\phi\}\} <^* \sigma_{j_\ell} \bigcup$$

$$\Big(\bigcup_{\substack{\alpha_r(y_h)=\{\phi\}\& \\ \alpha_r(y_k)=\{\phi\}}} \sigma_r \bigcup \{My_q : \alpha_{F(q)}(y_h) = \alpha_{F(q)}(y_k) = \{\phi\}\}\Big) \leq^* My_k$$

which completes the proof of Lemma 5.5.

Next we show that My_{j^*} is the least element of My_j in every well ordering $<^*$ of all sets satisfying R_3-R_5. Indeed we know that by (5.9)

$$My_j = \bigcup_{\alpha_t(y_j)=\{\phi\}} \sigma_t \bigcup \{My_k : \alpha_{F(k)}(y_j) = \{\phi\}\}$$

Now by (5.5.b) if

$$\alpha_t(y_j) = \{\phi\}$$

then

$$(\forall \alpha_\ell \in \Gamma)(\alpha_\ell(y_{j^*}) = \{\phi\} \rightarrow \ell < t)$$

This implies that if

$$\alpha_\ell(y_{j*})=\{\phi\} \text{ then } I_\ell \leq^* I_{t-1} <^* \{I_{t-1}\}$$

for every t such that $\alpha_t(y_i)=\{\phi\}$. Consequently if $\alpha_{F(q)}(y_{j*})=\{\phi\}$ then by lemma 5.6 $My_q <^* I_{F(q)} <^* \{I_{t-1}\}$ for every t such that $\alpha_t(y_j)=\{\phi\}$. Therefore every element of My_{j*} is less than $\{I_{t-1}\}$ for every t such that $\alpha_t(y_j)=\{\phi\}$. It follows by R_5 that

$$My_{j*} <^* \{\{I_{t-1}\}\} = I_t$$

for every t such that $\alpha_t(y_j)=\{\phi\}$. To complete our proof it remains to show that My_{j*} is less than or equal to every element My_k, with $\alpha_{F(k)}(y_j)=\{\phi\}$, in every well ordering $<^*$ satisfying R_3-R_5. Indeed if $\alpha_{F(k)}(y_j)=\{\phi\}$ then by (5.5.a) either $y_{j*} \sim_\Gamma y_k$ and $My_{j*}=My_k$ or $\bar{y}_{j*} < \bar{y}_k$ and by lemma 5.5 $My_{j*} <^* My_k$. We can then conclude that My_{j*} is the least element of My_j in any well ordering $<^*$ of all sets satisfying R_3,R_4, and R_5. Therefore $My_{j*}=\eta My_j$ and $My=M\bar{y}=My_{j*}=\eta My_j=\eta M\bar{x}=\eta Mx$. We have thus shown that M is indeed a model of Q_η independent of the particular choice of η completing the proof of our main theorem.

6. A Validity Test for a Weaker Theory

In this section we consider the theory which results by dropping the symbols $\bigcap, \bigcup, \backslash$ from the language considered in the preceding section. Moreover we assume that *only* restrictions R_0-R_3 must hold and that variables can range over *arbitrary* sets (not necessarily finite).

First we consider the case in which the η operator does not appear. So let Q be a conjunction of literals of type

$$(=,\neq)\ x = y,\ x \neq y$$

$$(\in,\notin)\ x \in y,\ x \notin y$$

where x,y are either variables or the constant ϕ.

We describe a satisfiability algorithm for Q originally given in [2]. Let $\sim$ be the smallest equivalence relation on the set of all the variables of Q such that $x=y$ *in* Q implies $x \sim y$. Choose a representative $\bar{x}$ in each equivalence class $\{y : y \sim x\}$, replace every variable by its

representative in Q and let $\bar{Q}$ be the resulting formula. The following is part of [2].

THEOREM 6.1. *Q has a model if and only if in $\bar{Q}$ the following conditions are satisfied.*

(6.1) *There is no explicit contradiction of the form $x \neq x$ or $x \in y$ & $x \notin y$.*

(6.2) *There is an ordering $y_1, y_2, \ldots, y_m$ of the variables of $\bar{Q}$ such that $y_1 \sim \phi$, and such that $y_i \in y_j$ in $\bar{Q}$ implies $i < j$.*

Let $x \hat{\in} y$ denote the fact that $x \in y$ is in $\bar{Q}$. If (6.1) and (6.2) are satisfied then models of Q can be built as follows. Choose sets $\sigma_j, j = 1, \ldots, m$, such that $\sigma_1 = \phi$. Next going upward in the ordering of variables, put

$$My_i = \sigma_i \bigcup \{My_j : y_j \hat{\in} y_i\} \tag{6.3}$$

and complete the definition of M by putting $Mx = M\bar{x}$ for every other variable x of Q. Then the following is true (see [2]).

THEOREM 6.2 *Formula (6.3) defines a model M of Q whenever the following conditions hold*

$$My_i \notin \sigma_j \text{ for } \textit{every } i,j = 1,\ldots,m \tag{6.4}$$

$$My_i \neq My_j \textit{ unless } i = j \tag{6.5}$$

Next let Q_η be a conjunction of literals of type

$(=, \neq)\, x = y, x \neq y$

$(\in, \notin)\, x \in y, x \notin y$

$(\eta)\, y_{j*} = \eta y_j$

Where x, y, y_j, y_{j*} are either variables or the constant ϕ. We want to show that the following holds.

THEOREM 6.3. *Either Q_η is unsatisfiable or there is an effectively constructible assignment of sets to variables which makes Q_η true independently of the particular choice of η subject only to satisfy restrictions $R_0 - R_3$*

In order to prove this theorem we make use of the following set theoretic lemma.

LEMMA 6.4 *For all sets $s_0, s_1, \cdots, s_{n+1}$* if

$$s_0 = s_{n+1} \text{ and } \eta s_j \in s_{j+1}, j=0,\ldots,n, \textit{ then } \eta s_0 = \eta s_1 = \cdots = \eta s_n$$

Proof. Indeed since $\eta s_j \in s_{j+1}$ then $\eta s_{j+1} \leq \eta s_j$ by R_2. This implies the lemma since $s_0 = s_{n+1}$.

To show that Theorem 6.3 holds, we first add to Q_η the following sentences.

$$\phi \in y_j \rightarrow y_{j*} = \phi \tag{6.6}$$

$$y_{j*} \in y_j \vee (y_j = \phi \,\&\, y_{j*} = \phi) \tag{6.7}$$

$$(y_{j_0} = y_{j_{n+1}} \,\&\, \mathop{\&}_{k=0}^{n} y_{j*_k} \in y_{j_{k+1}}) \rightarrow \mathop{\&}_{k=0}^{n+1} y_{j*_k} = y_{j*_o} \tag{6.8}$$

where $<y_j, y_{j*}>$ and $<y_{j_k}, y_{j*_k}>$ range over all pairs of variables appearing in literals of type (η).

For our purpose is then sufficient to show that, for each disjunct q_η in the disjunctive normal form of Q_η, either q_η is unsatisfiable or it has a model independent of η. To do this, let q_η be one of these disjuncts and let q be the result of dropping literals of type (η) in q_η. We can assume that no pair of equivalent non-identical variable exist in q. To finish the proof of Theorem 6.3 it is sufficient to demonstrate the following

LEMMA 6.5. *q_η has a model if and only if there is an ordering $z_1, z_2, \cdots, z_m$ of the variables of q_η such that*

$$z_1 \textit{ is (equivalent) } \phi; \tag{6.9}$$

$$z_u \hat{\in} z_v \rightarrow u < v; \tag{6.10}$$

$$(z_u \sim y_j \,\&\, z_v \sim y_{j*} \,\&\, z_w \hat{\in} z_u) \rightarrow v \leq w \tag{6.11}$$

for all variables z_u, z_v, z_w and all y_j, y_{j*} appearing in literals of type (η). Moreover in the positive

case a model of q_η *independent of the particular choice of* η *can be effectively constructed.*

Proof. First assume that q_η has a model M. For every pair x,y of variables of q_η define $x<^+y$ to mean that either

$$Mx<My \text{ in the well ordering of all sets associated with } \eta \tag{6.12}$$

or

$$Mx=My, x,y \text{ are distinct and for some } y_j, x\sim y_{j^*} \text{ and } y\hat{\in}y_j \tag{6.13}$$

Let us first prove that there are no cycles of $<^+$. That is there are no distinct variables $x_n,x_{n-1},\ldots,x_0$ of q_η such that $x_n<^+x_{n-1}<^+ \cdots <^+x_0$ where $n>0$ and x_0 is the same as x_n. Indeed by (6.12) and (6.13) this could only happen if for $k=n,n-1,\cdots,1$ $Mx_k=Mx_{k-1}$ and for some $y_{j_k}, x_k\sim y_{j^*_k}$ and $x_{k-1}\hat{\in}y_{j_k}$. Therefore we would have

$$x_0\hat{\in}y_{j_1}, x_1\sim y_{j^*_1}\hat{\in}y_{j_2}, \cdots, x_{n-1}\sim y_{j^*_{n-1}}\hat{\in}y_{j_n}$$

and since x_0 is x_n then $y_{j^*_n}\sim x_0$. It readily follows from (6.8) that all the x_i must be the same, contradicting $x_n<^+x_{n-1}$. Therefore the transitive closure of $<^+$ can be extended to a linear ordering $z_1,z_2,\cdots,z_n$ of the variables of q_η. Moreover if $Mx\neq\phi$ then $\phi<Mx$ in any well ordering of sets satisfying R_3. On the other hand if $Mx=\phi$ and for some $y_j, x\sim y_{j^*}$ and $\phi\hat{\in}y_j$ then by (6.6) x *is* ϕ. This shows that z_1 must be (equivalent to) ϕ , completing the proof of (6.9). As for (6.10) if $z_u\hat{\in}z_v$ then $Mz_u\in Mz_v, Mz_u<Mz_v$ and by (6.12) $u<v$. Finally if $z_u\sim y_j, z_v\sim y_{j^*}$ and $z_w\hat{\in}z_u$ then $Mz_v=\eta Mz_u$ and $Mz_w\in Mz_u$. It follows that $Mz_v\leq Mz_w$. Now, if z_v and z_w are not distinct then $v=w$. On the other hand if z_v,z_w are distinct and $Mz_v<Mz_w$ then by (6.12) $v<w$. Finally if z_v,z_w are distinct and $Mz_v=Mz_w$ then by (6.13) again $v<u$. This concludes the verification of (6.11) completing the proof of Lemma 6.5 in one direction.

Conversely, assume that there exists an ordering $z_1,z_2,\cdots,z_m$ of the variables of q_η

satisfying conditions (6.9) - (6.11). Then a model of q_η independent of η can be constructed as follows. Put $Mz_1=\sigma_1=\phi$ and going upward in the ordering of indices $i=2,3,\cdots,m$ put

$$\sigma_i=\{\{Mz_{i-1}\}\} \tag{6.14}$$

$$Mz_i=\sigma_i\bigcup\{Mz_j:z_j\hat{\in}z_i\} \tag{6.15}$$

By an easy induction on i it can be proven that

$$rank\ (Mz_i)=2(i-1)\text{ for } each\ i=1,2,\ldots,m. \tag{6.16}$$

This implies immediately that conditions (6.4) and (6.5) are satisfied. Therefore M is a model of q. To show that M is indeed a model of q_η let $z_v=\eta z_u$ be in q_η. Then for some j, $z_v\sim y_{j*}$ and $z_u\sim y_j$. By (6.7) either $z_v=z_u=\phi$ and $Mz_v=\phi=\eta\phi=\eta Mz_u$, or $z_v\hat{\in}z_u$ and by (6.15) $Mz_v\in Mz_u$. In this last case we want to show that Mz_v is the least element of Mz_u in any well ordering "<" of sets satisfying R_0-R_3. Indeed by (6.10) and (6.14) it follows that $Mz_v\in\cdots\in\{Mz_{u-1}\}$ which by R_3 yields $Mz_v<\{Mz_{u-1}\}$. Moreover if $Mz_w\in Mz_u$ with $z_w\hat{\in}z_u$ and z_w distinct from z_v then by (6.11) $u<w$. This by (6.14) and (6.15) gives $Mz_v\in\cdots\in\{Mz_{w-1}\}\in Mz_w$ which implies $Mz_v<Mz_u$. We can then conclude that Mz_v is the least element of Mz_u and that M is indeed a model of q_η. Therefore Lemma 6.5 is proved implying that Theorem 6.3 also holds.

7. Optimizations of the Weaker Validity Test

To improve the efficiency of the decision algorithm we have described, in forming Q_η we avoid to include in it the formulas (6.6) - (6.8). As before q_η denotes a disjunct of the disjunctive normal form of Q_η. However, we modify q_η as follows. We non-deterministically "guess" for which literals $y_{j*}=\eta y_j$ y_j will be nonempty in the model we are after. For each of these we add $y_{j*}\in y_j$ to q_η ; for the remaining literals $y_{k*}=\eta y_k$ we add $y_k=\phi$ and $y_{k*}=\phi$ to q_η. Define the relation $\sim$ on the variables of q_η as the smallest equivalence relation such that

$$x \sim y \text{ whenever } x=y \text{ is in } q_\eta \tag{7.1}$$

$$\phi \sim x \text{ whenever } z \in w \text{ is in } q_\eta \text{ with } z \sim \phi, x \sim y_{j*}, w \sim y_j. \tag{7.2}$$

For every "cycle"

$$z_0 \in x_1, z_1 \in x_2, \ldots, z_{n-1} \in x_n, z_n \in x_0 \text{ in } q_\eta \text{ with } n>0$$

$$\text{and } x_k \sim y_{j*_k}, z_k \sim y_{j_k} \text{ for } k=0,1,..,n, \text{ one must have } z_0 \sim z_1 \sim \cdots \sim z_m \tag{7.3}$$

The remaining steps of the validity test are the same as in the preceding algorithm. More precisely the existence of an ordering $z_1, z_2, \cdots, z_m$ of the variables of q_η satisfying (6.9) - (6.11) is tested. Finally Lemma 6.5 can be proved in analogy with the preceding proof.

8. Acknowledgments

The authors would like to thank Jacob T. Schwartz for suggesting the problem addressed in the present paper, and R. Parikh for helpful discussions and comments. The authors acknowledge partial support by the C.N.R. of Italy. This work was partially supported by NSF Grant # DCR-84-01633.

9. References

[1] Ferro, A., Omodeo, E.G., and Schwartz, J.T., "*Decision Procedures for Elementary Sublanguages of Set Theory.I. Multi-level Syllogistic and Some Extensions*", Comm. Pure Appl. Math. 33 , 533-608 (1980)

[2] Breban, M., Ferro, A., Omodeo, E.G., and Schwartz, J.T., "*Decision Procedures for Elementary Sublanguages of Set Theory. II. Formulas Involving Restricted Quantifiers, Together with Ordinal, Integer, Map and Domain Notions*", Comm. Pure. Appl. Math. 34, pp.177-185 (1981).

[3] Breban, M., and Ferro, A., *Decision procedures for elementary sublanguages of set theory. III. Restricted classes of formulas involving the powerset operator and the general set union*

operator. Advances in Appl. Math., 5, (1984).

[4] Cantone, D., Ferro, A., Micale, B., and Sorace, G., *Decision procedures for elementary sublanguages of set theory. IV. Formulae involving a rank operator and formulae involving one ocurence of the set operator* $\sum(x)=\{\{y\}|y\in x\}$, to appear in Comm. Pure Appl. Math.

[5] Cantone, D., Ferro, A., and Schwartz. J.T., *Decision procedures for elementary sublanguages of set theory. V. Multilevel syllogistic extended by the general union operator.* Journ. Comp. Syst. Sci., to appear.

[6] Ferro, A., *A note on the decidability of MLS extended by the powerset operator.* Comm. Pure Appl. Math. 38, 1985, pp.367-374.

[7] Cantone, D., Ferro, A., and Schwartz, J.T., *Decision procedures for elementary sublanguages of set theory. VI. Multilevel syllogistic extended by the powerset operator*, Comm. Pure Appl. Math. 38, 1985, pp.549-571.

[8] Leisenring, A.C., "Mathematical Logic and Hilbert's ϵ -Symbol", MacDonald & Co. (1969)

[9] Jech, T., "The Axiom of Choice" North Holland (1973)

[10] Jech, T., "Set Theory" Academic Press (1978)

[11] Bledsoe, W.W., *A Maximal Method for Set Variables in Automatic Theorem Proving*, Tech. Rept. ATP 33A, Automatic Theorem Proving Project, University of Texas, Austin, Texas.

NYU COMPSCI TR-226 c.1
Ferro, Alfredo
Decision procedures for
elementary sublanguages of
set theory. VII